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Large-*n* Limit of the Heisenberg Model: Random External Field and Random Uniaxial Anisotropy

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The thermodynamic equivalence of the large-n limit of the n-vector model in a random external field and the corresponding disordered spherical model is proved. An analytic expression for the free energy and a phase diagram of the large-n limit of the n-vector model with random uniaxial anisotropy are obtained by rigorous argument. The ferromagnetic order in the large-n limit is proved to be stable against the switching on of an arbitrarily small random anisotropy.

KEY WORDS: Random spin systems; ferromagnetic order.

1. The study of the thermodynamics of realistic spin models, as a rule, involves explicitly or implicitly the analysis of an infinite system of transcendental equations. In the ordered (translationally invariant) case, this system is reduced to a single equation if the number of components of the spin vector n tends to infinity.[1,2] That is why the thermodynamics of the limit model has been investigated in detail.[3]

One may expect that in the disordered case there will be no significant simplification in the large-n limit. Indeed, according to ref. 4, in the absence of the translational invariance, the large-n limit of the n-vector model appears to be the generalized spherical model. To study its behavior, one is to consider a macroscopic number of parameters satisfying an infinite system of equations. This problem is almost as complicated as the original one and its explicit solution is known only in the one-dimensional case.[5]

Nevertheless, there exist rather interesting disordered models for which the problem of the study of the thermodynamics in the large-n limit reduces

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to the solution of a single equation. In the present paper we consider two models of this kind:

(i) The classical n-vector ferromagnet in a random external field with the Hamiltonian

$$H_n = \frac{1}{2} \sum_{r,r'} J_{r-r'} \mathbf{s}_r \cdot \mathbf{s}_{r'} - \sum_r \mathbf{h}_r \cdot \mathbf{s}_r \tag{1}$$

$$\mathbf{s}_r = (s_r^1, \ldots, s_r^n); \qquad \sum_{\alpha=1}^{n} (s_r^\alpha)^2 = n; \qquad r, r \in V \subset \mathbb{Z}^d$$

where the field components h_r^α are independent identically distributed random variables for all $r \in \mathbb{Z}^d$, $\alpha = 1, \ldots, n$.

(ii) The classical n-vector ferromagnet with random anisotropy with the Hamiltonian

$$H_n = \frac{1}{2} \sum_{r,r'} J_{r-r'} \mathbf{s}_r \cdot \mathbf{s}_{r'} - n^{-1} \sum_r (\mathbf{D}_r \cdot \mathbf{s}_r)^2 \tag{2}$$

$$\mathbf{s}_r = (s_r^1, \ldots, s_r^n); \qquad \sum_{\alpha=1}^{n} (s_r^\alpha)^2 = n; \qquad r, r' \in V \subset \mathbb{Z}^d$$

where D_r are independent random vectors of a fixed length $\sum_{\alpha=1}^{n} (D_r^\alpha)^2 = n$ uniformly distributed over the respective sphere.

The modern idea of the critical behavior of the models (i)–(ii) is to a considerable extent based on the results obtained by approximations that are difficult to control.[6–9] The only known rigorous results concern the Ising model in a random field[10,11] and n-vector random-field model in dimensions $d \leqslant 4$.[11] The lack of rigorous results has stimulated the study of the large-n limit of these models[12–15]; however, this study used the replica trick as the main method.

Here we present a rigorous study of the thermodynamics of the models (i)–(ii) in the large-n limit. In Section 2 the large-n limit of the free energy of the model (i) is found on the assumption of the existence of the fourth moment of the external field. The answer involves only the first two moments of the field, does not depend on its distribution, and coincides with the free energy of the disordered spherical model in a random field.[16] The expression derived coincides also with that obtained in ref. 12 for the Gaussian random field by the replica trick.

In Section 3 it will be proved that if $\sigma < \sigma_c$, where

$$\sigma_c = \left[\frac{1}{2} \int_{[0,2\pi]^d} \frac{1}{\tilde{J}(0) - \tilde{J}(p)} \frac{dp}{(2\pi)^d} \right]^{-1/2}$$

[$\tilde{J}(p)$ is the Fourier transform of the interaction] depends only on the value of interaction and the dimension of the space, the large-n limit of the free energy of model (ii) coincides with that of the translationally invariant spherical model.[3] This means that if $\sigma < \sigma_c$, the random anisotropy does not affect the thermodynamics of the model, in particular, ferromagnetic order occurs at low temperatures (see Fig. 1). Such $n = \infty$ behavior differs from that which is expected for the finite-n model (ii)[7–9] and that was found by the replica trick for the large-n limit of a field-theoretic counterpart of model (ii).[12–15] When $\sigma > \sigma_c$ the second derivative of the large-n limit of the free energy as a function of temperature has a jump at the point $T = T_c$, where $kT_c = 2\sigma^2$. This singularity corresponds to the phase transition PM → D (see Figs. 1 and 2). In the phase D, the spins are frozen along the local anisotropy axes $\mathbf{D}_r$. A similar transition with the same critical temperature exists in the model of noninteracting spins with the Hamiltonian $H = -\sum_{r \in V} (\mathbf{D}_r \cdot \mathbf{s}_r)^2$ in the large-n limit.[17] In this case if n is finite, the PM → D phase transition disappears. It is natural to expect that there is no phase D for model (ii) when $n < \infty$.

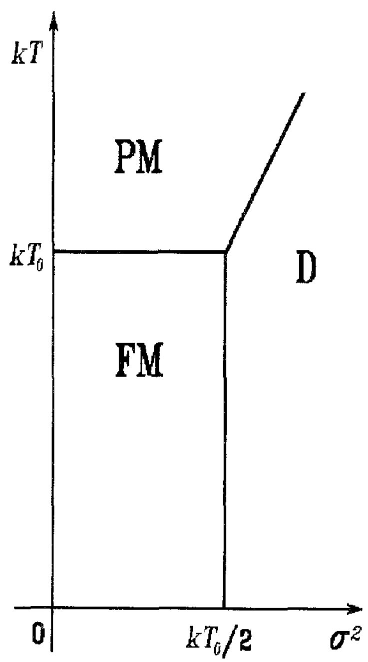

Fig. 1. Phase diagram of the large-n limit of the model (ii) with short-range interaction [$\tilde{J}(0) - \tilde{J}(p) \sim p^2$, $p \to 0$]. The space dimension $d \geqslant 3$; T_0 is the critical temperature in the translationally invariant case [$\sigma = 0$].

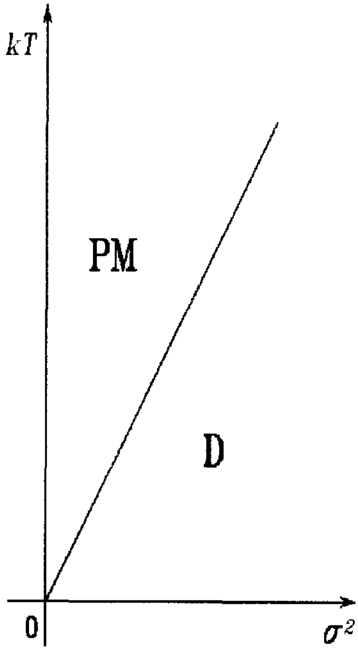

Fig. 2. Phase diagram of the large-n limit of the model (ii) with short-range interaction $[\tilde{J}(0)-\tilde{J}(p)\sim p^2,\ p\to 0]$. The space dimension $d=1, 2$.

2. In this section we derive an expression for the large-n limit of the free energy

$$f_{n,V}=-\frac{1}{n|V|\beta}\ln\int\exp(-\beta H_n)\prod_r dS_r \tag{3}$$

of model (i). Denote the average over the realizations of the external field h_r by $\mathbb{E}\{\cdot\}$. The following statement is the main result of this section.

Theorem 1. Let us assume that $\sum_{r\in\mathbb{Z}^d}J_r$ and $\mathbb{E}\{(h_r^\alpha)^4\}$ are finite. Then,

$$\lim_{n\to\infty}\lim_{|V|\to\infty}\mathbb{E}\{f_{n,V}\}$$

$$=\max_{z>\tilde{J}(0)}\left[\frac{1}{2\beta}\int_{[0,2\pi]^d}\ln[z-\tilde{J}(p)]\frac{dp}{(2\pi)^d}-\frac{\sigma^2}{2}\int_{[0,2\pi]^d}\frac{1}{z-\tilde{J}(p)}\frac{dp}{(2\pi)^d}\right.$$
$$\left.-\frac{h^2}{2[z-\tilde{J}(0)]}-\frac{z}{2}-\frac{1}{2\beta}\ln\frac{2\pi}{\beta}\right] \tag{4}$$

where h and σ^2 are the mean value and the variance of the external field, respectively, $h = \mathbb{E}\{h_r^\alpha\}$, $\sigma^2 = \mathbb{E}\{(h_r^\alpha - h)^2\}$, and $\tilde{J}(p)$ is the Fourier transform of the interaction.

Remarks. 1. According to ref. 18, the free energy in the thermodynamic limit is self-averaging. This means that for almost all realizations of the external field

$$\lim_{|V| \to \infty} f_{n,V} = \lim_{|V| \to \infty} \mathbb{E}\{f_{n,V}\}$$

Therefore the average over the realizations of the field $\mathbf{h}_r$ in the lhs of (4) may be omitted.

2. The rhs of expression (4) coincides with the free energy of the spherical model in a random external field.[16] For definiteness, we confine ourselves to the short-range interaction $[\tilde{J}(0) - \tilde{J}(p) \sim p^2,\ p \to 0]$. The analysis of the rhs of (4) shows[16] that in a space of dimension $d = 3, 4$, an arbitrarily small random field with a zero mean destroys the ferromagnetic order. If $d \geqslant 5$ and the fluctuations of the external field $\mathbf{h}_r$ are not too large,

$$\sigma < \sigma_c = \left| \int_{[0,2\pi]^d} \frac{1}{[\tilde{J}(0) - \tilde{J}(p)]^2} \frac{dp}{(2\pi)^d} \right|^{-1/2}$$

then it only suppresses, but does not destroy, the ferromagnetic order. Namely, at temperatures $T < T_c = T_0[1 - (\sigma/\sigma_c)^2]$, where T_0 is the critical temperature of the translationally invariant spherical model,[3]

$$kT_0 = \left[\int_{[0,2\pi]^d} \frac{1}{\tilde{J}(0) - \tilde{J}(p)} \frac{dp}{(2\pi)^d} \right]^{-1} \tag{5}$$

the spontaneous magnetization is nonzero and is $|(T_c - T)/T_0|^{1/2}$.

Proof. We use a general approach to the calculation of the large-n limit of the free energy of spin systems proposed in ref. 2. Namely, consider the spherical model with the Hamiltonian

$$H_n^s = -\frac{1}{2} \sum_{r,r'} J_{r-r'} \mathbf{x}_r \cdot \mathbf{x}_{r'} - \sum_r \mathbf{h}_t \cdot \mathbf{x}_r + \frac{z}{2} \sum_r (\mathbf{x}_r \cdot \mathbf{x}_r - n) \tag{6$'$}$$

where $\mathbf{x}_r \in \mathbb{R}^n$, and the constant z is the solution of the equation

$$\mathbb{E}\left\{ |V|^{-1} \sum_{r \in V} \langle \mathbf{x}_r \cdot \mathbf{x}_r \rangle_{H_n^s} \right\} = n \tag{6$''$}$$

Evidently, equality (4) will be true if we prove that in the limit $n = \infty$ the difference $\mathbb{E}\{f_{n,V} - f_{n,V}^s\}$ tends to zero uniformly in V. To compare the free energies corresponding to (1) and (6), we introduce, following ref. 2, an intermediate Hamiltonian $\Gamma(\mathbf{x})$. In the polar coordinates $\mathbf{x}_r = \rho_r \cdot s_r$, $\rho_r = n^{-1}\mathbf{x}_r \cdot \mathbf{x}_r$,

$$\Gamma(\mathbf{x}) = H_n + n \sum_r (\rho_r - 1) m_r + Bn \sum_r (\rho_r - 1)^2 + \sum_r \Phi_n(m_r)$$

where B is a constant which we shall choose later,

$$m_r = n^{-1} \sum_{r'} J^z_{r-r'} \mathbf{s}_r \cdot \mathbf{s}_{r'} - n^{-1} \mathbf{h}_r \cdot \mathbf{s}_{r'}, \qquad J^z_{r-r'} = z\delta_{r-r'} - J_{r-r'} \tag{7}$$

$$\Phi_n(m_r) = \beta^{-1} \ln \left(n^{1/2} \int_0^\infty \exp\{-\beta[n(\rho - 1) m_r + nB(\rho - 1)^2]\} \rho^{n-1}\, d\varphi \right)$$

The function $\Phi_n(m_r)$ in the expression for $\Gamma(\mathbf{x})$ is chosen in such a way that

$$\int \exp[-\beta \Gamma(\mathbf{x})] \prod_r dx_r = \int \exp(-\beta H_n) \prod_r dS_r \tag{8}$$

and hence the free energies corresponding to H_n and $\Gamma(\mathbf{x})$ coincide.

Simple calculations based on the Jensen and Cauchy inequalities and $\ln \rho \leqslant \rho - 1$ show that

$$-(2\beta)^{-1} \ln n + \underline{\varepsilon}(\mathbf{h}_r) \leqslant \Phi_n(m_r) \leqslant (4\beta B)^{-1}\, n(1 - \beta m_r)^2 + \bar{\varepsilon}(\mathbf{h}_r) \tag{9}$$

and moreover,

$$\mathbb{E}\{\underline{\varepsilon}(\mathbf{h}_r)\} \geqslant \underline{C}, \qquad \mathbb{E}\{\bar{\varepsilon}(\mathbf{h}_r)\} \leqslant \bar{C}$$

where $\underline{C}$ and $\bar{C}$ do not depend on V on n.

Let $R = \Gamma(\mathbf{x}) - H_n^s$. The Bogolubov inequality[19] together with (8) imply

$$(n\,|V|)^{-1} \langle R \rangle_\Gamma \leqslant f_{n,V} - f_{n,V} \leqslant (n\,|V|)^{-1} \langle R \rangle_{H_n^s} \tag{10}$$

In order to estimate $\langle R \rangle_\Gamma$, we note that

$$R = -\frac{1}{2} \sum_{r,r'} J^z_{r-r'} (\rho_1 - 1)(\rho_{r'} - 1) \mathbf{s}_r \cdot \mathbf{s}_{r'} + Bn \sum_r (\rho_r - 1)^2 + \sum_r \Phi_n(m_r)$$

Thus, if we choose B equal to the l_2-norm of the matrix $J^z_{r-r'}$, then

$$R \geqslant Bn/2 \sum_r (\rho_r - 1)^2 + \sum_r \Phi_n(m_r) \geqslant \sum_r \Phi_n(m_r)$$

Hence,

$$f_{n,V}-f^s_{n,V}\geqslant -(2\beta n)^{-1}\ln n+(n|V|)^{-1}\sum \underline{\varepsilon}(\mathbf{h}_r) \tag{11}$$

$$\mathbb{E}\{f_{n,V}-f^s_{n,V}\}\geqslant -(2\beta n)^{-1}\ln n+n^{-1}\underline{C} \tag{12}$$

To estimate the rhs of (10) from the above, we need the inequality

$$n^{-2}|\mathbb{E}\{\langle \mathbf{x}_{r_1}\cdot\mathbf{x}_{r_2}\mathbf{x}_{r_3}\cdot\mathbf{x}_{r_4}\rangle_{H^s_n}\} - \mathbb{E}\{\langle \mathbf{x}_{r_1}\cdot\mathbf{x}_{r_2}\rangle_{H^s_n}\}\,\mathbb{E}\{\langle \mathbf{x}_{r_3}\cdot\mathbf{x}_{r_4}\rangle_{H^s_n}\}|\leqslant n^{-1}C \tag{13}$$

which may be easily verified, if $\mathbb{E}\{(h^\alpha_r)^4\}<+\infty$. Since $B=\|J^z\|_2$, we have

$$R\leqslant 3Bn/2\sum_r(\rho_r-1)^2+\sum_r\Phi_n(m_r)$$

and (9) implies

$$\mathbb{E}\{(n|V|)^{-1}\langle R\rangle_{H^s_n}\}\leqslant 3B/(2|V|)\sum_r\mathbb{E}\{\langle(\rho_r-1)^2\rangle_{H^s_n}\} + (4\beta|V|B)^{-1}\sum_r\mathbb{E}\{\langle(1-\beta m_r)^2\rangle_{H^s_n}\}+n^{-1}\bar{C} \tag{14}$$

Let us recall that z is the solution of (6″) and $\rho_r=n^{-1}\mathbf{x}_r\cdot\mathbf{x}_r$. Thus, by using the inequality $(\rho-1)^2\leqslant(\rho^2-1)^2$ which is obvious if $\rho>0$, and factorizing the average $\mathbb{E}\{\langle\cdot\rangle\}$ by (13), we obtain

$$|V|^{-1}\sum_r\mathbb{E}\{\langle(\rho_r-1)^2\rangle_{H^s_n}\}\leqslant n^{-1}C$$

In order to estimate the second term on the rhs of (14), we use the definition of m_r:

$$(1-\beta m_r)^2\leqslant 2\left[1-\beta n^{-1}\sum_{r'}J^z_{r-r'}\mathbf{x}_r\cdot\mathbf{x}_{r'}-\beta n^{-1}\mathbf{h}_r\cdot\mathbf{x}_r\right]^2 + 4\beta^2n^{-2}\left[\sum_{r'}J^z_{r-r'}(\rho_r\rho_{r'}-1)\mathbf{s}_r\cdot\mathbf{s}_{r'}\right]^2+4[\beta(1-\rho_r)(\mathbf{h}_r\cdot\mathbf{s}_r)/n]^2$$

Let us denote the average $\mathbb{E}\{\langle\cdot\rangle_{H^s}\}$ of the three terms on the rhs of the later inequality by I^i_r, $i=1, 2, 3$, respectively. In view of the identity

$$\beta\left\langle\sum_{r'}J^z_{r-r'}\mathbf{x}_r\cdot\mathbf{x}_{r'}-\mathbf{h}_r\cdot\mathbf{x}_r\right\rangle_{H^s_n}=n\delta_{r-r'}$$

we have

$$I_r^1 = \mathbb{E}\left\{\left\langle\left[\beta n^{-1}\sum_{r'} J^z_{r-r'}\mathbf{x}_r\cdot\mathbf{x}_{r'} - \mathbf{h}_r\cdot\mathbf{x}_r\right]^2\right\rangle_{H_n^s}\right\} - 1 \tag{15}$$

Now, by using (13), we get $I_r^1 < n^{-1}C$. We estimate I_r^2 and I_r^3 in a similar way. Therefore,

$$\mathbb{E}\{(n|V|)^{-1}\langle R\rangle_{H_n^s}\} \leqslant n^{-1}\hat{C}$$

and by inequality (10),

$$n^{-1}\underline{C} - (2\beta n)^{-1}\ln n \leqslant \mathbb{E}\{f_{n,V} - f^s_{n,V}\} \leqslant n^{-1}\hat{C} \tag{16}$$

where $\hat{C}$ and $\underline{C}$ do not depend on V. The theorem is proved.

3. In this section section we find the large-n limit of the free energy of an n-vector ferromagnet with random uniaxial anisotropy.

Theorem 2. Let us suppose that $\sum_{r\in\mathbb{Z}^d} J_r < \infty$. Then, the free energy $f_{n,V}$ corresponding to the Hamiltonian

$$H_n = -\frac{1}{2}\sum_{r,r'} J_{r-r'}\mathbf{s}_r\cdot\mathbf{s}_{r'} - n^{-1}\sum_r (\mathbf{D}_r\cdot\mathbf{s}_r)^2 - h\sum_r\sum_\alpha s_r^\alpha \tag{17}$$

satisfies the equality

$$\lim_{n\to\infty}\lim_{|V|\to\infty}\mathbb{E}\{f_{n,V}\} = \lim_{|V|\to\infty}\min_{c\in\mathbb{R}}\max_{z>\tilde{J}(0)} F_V(z,c) \tag{18}$$

where

$$F_V(z,c) = \frac{1}{2|V|\beta}\sum_{p\in V^*}\ln[z-\tilde{J}(p)] - \frac{h^2}{2[z-\tilde{J}(0)]} - \frac{z}{2} + c^2\left(1 - \frac{2\sigma^2}{|V|}\sum_{p\in V^*}\frac{1}{z-\tilde{J}(p)}\right) - \frac{1}{2\beta}\ln\frac{2\pi}{\beta} \tag{19}$$

The proof of the theorem is based on the observation that in the large-n limit, random uniaxial anisotropy may be replaced by an "effective" external field. Here is the corresponding statement.

Lemma. Let $f_{n,V}$ be the free energy of model (17) and $f^a_{n,V}$ be the free energy corresponding to the Hamiltonian

$$H_n^a = H_n + n^{-1}\sum_r (\mathbf{D}_r\cdot\mathbf{s}_r - na_r)^2 = \frac{1}{2}\sum_{r,r'} J_{r-r'}\mathbf{s}_r\cdot\mathbf{s}_{r'} - \sum_r\sum_\alpha (h + 2a_r D_r^\alpha)s_r^\alpha + n\sum_r a_r^2 \tag{20}$$

in which a_r are real. Then

$$0 \leqslant \min_{a \in \mathbb{R}^{|V|}} f^a_{n,V} - f_{n,V} \leqslant [2\sigma^2/(\beta n)]^{1/2}$$

Proof. We use the well-known approximating Hamiltonians method[20] in the form proposed in ref. 21. Denote

$$H^a_n(\varepsilon) = H^a_n + \sum_r \varepsilon_r \mathbf{D}_r \cdot \mathbf{s}_r, \qquad H_n(\varepsilon) = H_n + \sum_r \varepsilon_r \mathbf{D}_r \cdot \mathbf{s}_r$$

According the Bogolubov inequality,[19] for $H^a_n(\varepsilon)$ and $H_n(\varepsilon)$, we have

$$(n^2 |V|)^{-1} \sum_r \langle (\mathbf{D}_r \cdot \mathbf{s}_r - n a_r)^2 \rangle_{H^a_n(\varepsilon)} \leqslant f^a_{n,V}(\varepsilon) - f_{n,V}(\varepsilon)$$

$$\leqslant (n^2 |V|)^{-1} \sum_r \langle (\mathbf{D}_r \cdot \mathbf{s}_r - n a_r)^2 \rangle_{H_n(\varepsilon)}$$

Hence,

$$0 \leqslant \min_{a \in \mathbb{R}^{|V|}} f^a_{n,V}(\varepsilon) - f_{n,V}(\varepsilon) \leqslant (n^2 |V|)^{-1} \sum_r \langle (\mathbf{D}_r \cdot \mathbf{s}_r - \langle \mathbf{D}_r \cdot \mathbf{s}_r \rangle)^2 \rangle_{H_n(\varepsilon)}$$

Denote the minimum point of the lhs of the latter inequality by $a(\varepsilon)$. Then

$$f^{a(\varepsilon)}_{n,V}(\varepsilon) \leqslant f_{n,V}(\varepsilon) - (\beta n)^{-1} \sum_r \frac{\partial^2 f_{n,V}(\varepsilon)}{\partial \varepsilon_r^2}$$

Multiply this inequality by the Green function of the operator $I - (\beta n)^{-1} \sum_{r \in V} \partial^2/\partial \varepsilon_r^2$, and the equation

$$\delta(\varepsilon) = G(\varepsilon) - (\beta n)^{-1} \sum_r \frac{\partial^2 G(\varepsilon)}{\partial \varepsilon_r^2}$$

which holds for $G(\varepsilon)$, by $f_{n,V}(\varepsilon)$. Subtract the second obtained expressed from the first one and integrate the difference. Since

$$\int \left[f_{n,V}(\varepsilon) \frac{\partial^2 G(\varepsilon)}{\partial \varepsilon_r^2} - G(\varepsilon) \frac{\partial^2 f_{n,V}(\varepsilon)}{\partial \varepsilon_r^2} \right] \prod_r d\varepsilon_r = 0$$

we get the inequality

$$f_{n,V} = f_{n,V}(\varepsilon)|_{\varepsilon = 0} \geqslant \int G(\varepsilon) f^{a(\varepsilon)}_{n,V} \prod_r d\varepsilon_r$$

Hence,

$$0 \leqslant \min_{a \in \mathbb{R}^{|V|}} f^a_{n,V} - f_{n,V} \leqslant f^{a(0)}_{n,V}(0) - \int G(\varepsilon) f^{a(\varepsilon)}_{n,V} \prod_r d\varepsilon_r$$

$$\leqslant \int G(\varepsilon)[f^{a(0)}_{n,V}(\varepsilon) - f^{a(\varepsilon)}_{n,V}(\varepsilon)] \prod_r d\varepsilon_r$$

$$\leqslant \int G(\varepsilon) \|\varepsilon\| \, \|\text{grad}\, f^{a(\varepsilon)}_{n,V}\| \prod_r d\varepsilon_r$$

Now, by using the inequality

$$\|\text{grad}\, f^{a(\varepsilon)}_{n,V}\|^2 \leqslant (n|V|)^{-2} \sum_r \langle \mathbf{D}_r \cdot \mathbf{s}_r \rangle^2_{H_n(\varepsilon)} \leqslant \sigma^2 |V|^{-1}$$

we obtain

$$\begin{aligned} 0 &\leqslant \min_{a \in \mathbb{R}^{|V|}} f^a_{n,V} - f_{n,V} \\ &\leqslant |V|^{-1/2} \sigma \int G(\varepsilon) \, \|\varepsilon\| \prod_r d\varepsilon_r \\ &\leqslant \left[|V|^{-1} \sigma^2 \int G(\varepsilon) \, \|\varepsilon\|^2 \prod_r d\varepsilon_r \right]^{1/2} = \frac{\sigma^2}{n\beta |V|} \sum_{r,r'} \int \frac{\partial^2 G(\varepsilon)}{\partial \varepsilon_r^2} \varepsilon_{r'}^2 \prod_r d\varepsilon_r \\ &\leqslant \left(\frac{2\sigma^2}{\beta n} \right)^{1/2} \end{aligned}$$

The lemma is proved.

Since the "effective" external field $\mathbf{h}^{\text{eff}}_r = 2a_r \mathbf{D}_r$ is inhomogeneous, we cannot apply directly the results of the preceding section to prove Theorem 2. We need the following arguments. In view of the inequality

$$\mathbb{E}\{\min_{a \in \mathbb{R}^{|V|}} f^a_{n,V}\} \leqslant \min_{c \in \mathbb{R}} \mathbb{E}\{f^a_{n,V}|_{a_r = c}\}$$

inequality (16) and the lemma yield

$$\mathbb{E}\{f_{n,V}\} \leqslant \min_{c \in \mathbb{R}} \max_{z > \mathcal{J}(0)} F_V(z, c) + O(1/n)$$

where $F_V(z, c)$ is defined by (19).

On the other hand,

$$\min_{a \in \mathbb{R}^{|V|}} f^a_{n,V} = \min_{a \in [-\sigma, \sigma]^{|V|}} f^a_{n,V}$$

Let us fix $a \in [-\sigma, \sigma]^{|V|}$ and construct the spherical counterpart of the Hamiltonian (20) [cf. (6′), (6″)],

$$H_n^{a,s} \doteq -\frac{1}{2}\sum_{r,r'} J_{r-r'}\mathbf{x}_r \cdot \mathbf{x}_{r'} - \sum_r \sum_\alpha (h + 2a_r D_r^\alpha) x_r^\alpha + \frac{z}{2}\sum_r (\mathbf{x}_r \cdot \mathbf{x}_r - n) + n\sum_r a_r^2$$

Here z is the solution of the equation

$$\mathbb{E}\left\{\left\langle \frac{1}{n|V|}\sum_r \mathbf{x}_r \cdot \mathbf{x}_r \right\rangle_{H_n^{a,s}}\right\} = 1$$

or

$$\frac{1}{|V|}\sum_{p \in V^*} \frac{1}{z - \tilde{J}(p)} + \frac{2\sigma^2}{|V|}\sum_{p \in V^*} \frac{1}{[z - \tilde{J}(p)]^2} \frac{1}{|V|}\sum_r a_r^2 + \frac{h^2}{z - \tilde{J}(0)} = 1 \qquad (21)$$

It is easy to see that for $H_n^{a,s}$ the inequality (11) still holds and moreover for the external field $h_r^\alpha = h + 2a_r D_r^\alpha$ the constants ε_r may be chosen nonrandom and independent either of r or of $a \in [-\sigma, \sigma]^{|V|}$. Therefore,

$$f_{n,V}^a \geqslant f_{n,V}^{a,s} + O\left(\frac{\ln n}{n}\right)$$

Denote the difference $f_{n,V}^{a,s} - \mathbb{E}\{f_{n,V}^{a,s}\}$ by Δf. Simple calculations show that

$$|\Delta f| \leqslant \frac{1}{|V|}\sum_{p \in V^*} \frac{1}{z - \tilde{J}(p)} \frac{1}{|V|}\sum_{r,r'} \left| \sigma^2 \delta_{r-r'} - n^{-1}\sum_\alpha D_r^\alpha D_{r'}^\alpha \right| + \frac{2\sigma^2}{|V|}\sum_r \left| n^{-1}\sum_\alpha D_r^\alpha \right|$$

and

$$\mathbb{E}\{|\Delta f|\} \leqslant \frac{2\sigma^4}{n^{1/2}[z - \tilde{J}(0)]} + \frac{2\sigma^2}{n^{1/2}}$$

Since z is the solution of (21), we have $z - J(0) \geqslant |h|$ and hence $\mathbb{E}\{|\Delta f|\} \leqslant C/n^{1/2}$. Thus,

$$\mathbb{E}\{f_{n,V}^a\} \geqslant \mathbb{E}\{f_{n,V}^{a,s}\} - \mathbb{E}\{\Delta f\} + O\left(\frac{\ln n}{n}\right) \geqslant \mathbb{E}\{f_{n,V}^{a,s}\} + O\left(\frac{\ln n}{n}\right)$$

The latter inequality and the lemma imply in turn that

$$\mathbb{E}\{f_{n,V}\} \geqslant \min_{a \in [-\sigma,\sigma]^{|V|}} \mathbb{E}\{f_{n,V}^{a,s}\} + O\left(\frac{\ln n}{n}\right)$$

But we have

$$\min_{a \in [-\sigma,\sigma]^{|V|}} \mathbb{E}\{f_{n,V}^{a,s}\} = \lim_{a \in [-\sigma,\sigma]^{|V|}} \max_{z > \tilde{J}(0)} F_V\left(z, \frac{1}{|V|} \sum_r \alpha_r^2\right)$$

where $F_V(z, c)$ is defined by (19). Therefore,

$$\mathbb{E}\{f_{n,V}\} \geqslant \min_{c \in \mathbb{R}} \max_{z > \tilde{J}(0)} F_V(z, c) + O\left(\frac{\ln n}{n}\right)$$

Theorem 2 is proved.

By using this theorem, one can easily find the large-n limit of the free energy of model (ii). We omit the simple calculations and give only the final result. Denote the critical temperature of the translationally invariant spherical model by T_0 [see Eq. (5)], and let f^s be the free energy of the model, i.e.

$$f^s = \max_{z \geqslant \tilde{J}(0)} F^0(z)$$

where

$$F^0(z) = \frac{1}{2\beta} \int_{[0,2\pi]^d} \ln[z - \tilde{J}(p)] \frac{dp}{(2\pi)^d} - \frac{z}{2} - \frac{1}{2\beta} \ln \frac{2\pi}{\beta} \tag{22}$$

The minimum with respect to c in (18) is at $c = 0$ and the large-n limit of the free energy of model (ii) coincides with f^s for all temperatures if $\sigma \leqslant \sigma_c$, where $\sigma_c = (kT_0/2)^{1/2}$, and for sufficiently large temperatures ($kT > 2\sigma^2$) if $\sigma > \sigma_c$, i.e.,

$$\lim_{n \to \infty} \lim_{|V| \to \infty} \mathbb{E}\{f_{n,V}\} = f^s$$

Hence, in the large-n limit, the ferromagnetic order is stable against the switching on of random anisotropy.

If $\sigma > \sigma_c$ and $kT < 2\sigma^2$, the above-mentioned minimum is reached at $c = c^*$, and

$$\lim_{n \to \infty} \lim_{|V| \to \infty} \mathbb{E}\{f_{n,V}\} = F^0(z^*)$$

where $F^0(z)$ is defined by (22), and z^* is the solution of the equation

$$2\sigma^2 = \int_{[0,2\pi]^d} \frac{1}{z - \tilde{J}(p)} \frac{dp}{(2\pi)^d}$$

Thus, if $\sigma > \sigma_c$, the phase transition to the low-temperature phase D occurs under the temperature $T_c = 2\sigma^2$. In this phase, which exists in a space of an arbitrary dimension *d*, the spins are frozen along the anisotropy local axes $\mathbf{D}_r$. Here we do not discuss the question of the convergence of $\eta^{-1}\langle \mathbf{D}_r \cdot \mathbf{s}_r \rangle$ to the parameter c^*. This will be done elsewhere. Note, however, that a random external field $\varepsilon \mathbf{D}_r$ with an arbitrarily small ε destroys the above transition. Besides, according to ref. 17, the same phase transition takes place in the large-*n* limit of a system of noninteracting spins $[H = -\sum_{r \in V} (\mathbf{D}_r \cdot \mathbf{s}_r)^2]$, in which there are no phase transitions in this case if *n* is finite.

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